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Heat Content Asymptotics - Theory and Practice

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Abstract: The heat content asymptotics are a short time measure of the total heat content of a solid in $\mathbb{R}^3$ which have been studied extensively in the mathematical literature. We propose a series of experiments to determine the extent to which the mathematical theories describe physical reality. This is joint work with B. Boggs and S. Espy.

Keywords: Heat content asymptotics, Dirichlet and Robin boundary conditions, interferometric measurements, CAMCOR focused ion beam.

MSC 2010: 58J32; 58J35; 35K20

1 Introduction

Let $\Delta = -\partial^2_{x^1} - \cdots - \partial^2_{x^m}$ be the Laplacian on a solid $M \subset \mathbb{R}^3$. Consider the heat equation

$\partial_t u + \Delta u = 0$ (Evolution equation)

$\lim_{t\downarrow 0} u(\cdot;t) = \phi(\cdot)$ (Initial condition)

$\mathscr{B}u = 0$ (Boundary condition)

Here $\mathscr{B}$ is a suitable description of what happens near the boundary. Typical examples are

$\mathscr{B}_D u = u|_{\partial M}$ (Dirichlet boundary conditions)

$\mathscr{B}_R u = (\partial_\nu u + Su)|_{\partial M}$ for ν the inward unit normal. (Robin boundary conditions)

If (M,g) is a smooth bounded domain in $\mathbb{R}^3$, then Dirichlet boundary conditions correspond to dropping the body into ice-water. The boundary is instantaneously cooled to 0C. For Robin boundary conditions, the heat flow across the boundary is proportional to the temperature of the boundary; again the exterior is held at 0C. Neumann boundary conditions correspond to $S = 0$; there is no heat flow across the boundary - the boundary is perfectly insulated. If ρ is the specific heat, then the total heat energy content is $\beta(\phi,\rho,D,\mathscr{B})(t) := \int_M u(x;t)\rho(x)dx$. Assume the solid is in equilibrium at $t = 0$ so $\phi = \kappa$ is constant. For Dirichlet or Robin boundary conditions, $\beta \sim \beta_0 + \beta_1 t^{1/2} + \beta_2 t + \beta_3 t^{3/2} + \beta_4 t^2 + \ldots$ for $\beta_0 = \kappa\int_M \rho dx$ where the β_i are locally computable. Let L be the second fundamental form.

1. Dirichlet boundary conditions:
   (a) $\beta_1 = -\frac{2}{\sqrt{\pi}}\kappa\int_{\partial M}\rho dy$, (b) $\beta_2 = \kappa\int_{\partial M}\{\frac12 L_{aa}\rho - \rho_{;m}\}dy$.
   (c) $\beta_3 = \kappa\int_{\partial M}\{\frac23\rho_{;mm} - \frac23 L_{aa}\rho_{;m} + \frac1{12}L_{aa}L_{bb} - \frac16 L_{ab}L_{ab}\}\rho dy$.
2. Robin boundary conditions:
   (a) $\beta_1 = 0$. (b) $\beta_2 = \kappa\int_{\partial M} S\rho dy$.
   (c) $\beta_3 = \frac23\cdot\frac{2}{\sqrt{\pi}}\kappa\int_{\partial M} S(\partial_\nu + S)\rho dy$. (d) $\beta_4 = \kappa\int_{\partial M}\{-\frac12 S\Delta\rho + (\frac12 S + \frac14 L_{aa})S(\partial_\nu + S)\rho\}dy$.

For Dirichlet boundary conditions, $\beta(t) \sim \beta_0 - \kappa \frac{2}{\sqrt{\pi}} t^{1/2} \operatorname{vol}(\partial M) + O(t)$ yields a power law in $t^{1/2}$ for the cooling. With Robin boundary conditions, $\beta(t) \sim \beta_0 + tS\kappa \operatorname{vol}(\partial M) + O(t^{3/2})$ yields a power law in t for the cooling (S will in general be negative).

A 19th centure measurement 10^{-3} **sec**

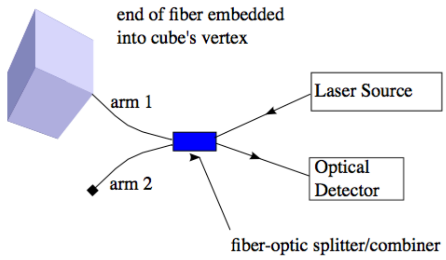
• Construct test shapes out of brass and aluminum in the shape of a sphere, a torus, a cube, and a cylinder. Insert probes into different parts of the solids. In the case of a cube, for example, the center of a face, the center of an edge, and a vertex are obvious points. One heats the solid to uniform temperature κ and then immerses it suddenly in ice water. One is interested in knowing the power law controlling the temperature decay for short time. And if the power is not $t^{1/2}$ or t controlling the cooling, then one knows there is a fractal phenomena occurring.

20th century – Interferometric Measurements 10^{-7} **sec.**

• Interferometer arm 1: A length of fiber-optic cable at least partly embedded within an object's region of interest (e.g., the vertex of a cube).

• Interferometer arm 2: An equal length fiber-optic cable.

• Measure the resulting time-dependent interference patterns caused by, for example, the hot object's contraction when brought into contact with a cold bath.



Evenly split coherent light from the laser source travels down arms 1 and 2 reflecting back off the fiber ends. It is then recombined and the resultant interference is subsequently detected.

A 21st century measurement

• With CAMCOR's focused Ion Beam machine make micron-sized objects (cube, torus, etc.) from a material that is transparent at wavelength λ-trap and absorbing at wavelength λ-heat. Employ the APL's optical tweezers apparatus to spatially trap and heat the object. Use the resultant fast-time decay of the object's Brownian motion as a measure of the transient heat flow from the object to its surrounding liquid bath.

References

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