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# Analytic Family of Solution Operators for Degenerate Fractional Equations

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**Abstract:** We study a differential equation in a Banach space with degenerate operator at the fractional derivative. Degenerate analytic family of solution operators are found and in a case of reflexive Banach spaces unique solvability of the Cuachy problem for the equation is proved.

**Keywords:** fractional derivative; degenerate evolution equation; semigroup theory; analytic family of solution operators.

**MSC 2010:** 47D06; 47D09; 34G10; 26A33.

## 1 Introduction

Consider the Cauchy problem

$$u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, m-1, \tag{1}$$

for a fractional differential equation

$$D_t^\alpha Lu(t) = Mu(t), \quad t > 0, \tag{2}$$

with linear closed operators $L$ and $M$ that are densely defined in a Banch space $\mathfrak{U}$ on $D_L$ and $D_M$ correspondingly, acting to $\mathfrak{V}$. Here $D_t^\alpha$ is the Caputo fractional derivative with $\alpha > 0$, $m$ is a smallest integer greater than or equal to $\alpha$. Denote the fractional integral by $J_t^\alpha$.

The feature of the equation is a nontrivial kernel $\ker L$ of the operator $L$: $\ker L \neq \{0\}$. Such equations will be called as degenerate. The conditions is studied for a unique solution existence of problem (1), (2). Analytic family of solution operators is constructed and it is shown that solutions belong to a subspace of $\mathfrak{U}$. This work is a continuation of [1–3].

## 2 Main results

For Banach spaces $\mathfrak{U}$, $\mathfrak{V}$ the Banach space of linear continuous operators from $\mathfrak{U}$ to $\mathfrak{V}$ will be denoted by $\mathscr{L}(\mathfrak{U};\mathfrak{V})$. If $\mathfrak{V} = \mathfrak{U}$, then this denotation will have a form $\mathscr{L}(\mathfrak{U})$. The set of complex numbers $\mu \in \mathbb{C}$ such that $(\mu L - M)^{-1} L \in \mathscr{L}(\mathfrak{U})$ and $L(\mu L - M)^{-1} \in \mathscr{L}(\mathfrak{V})$ is denoted by $\rho^L(M)$.

We consider the following conditions:

(I) there exist such constants $a_0 > 0$ and $\theta_0 \in (\pi/2, \pi)$ that for all

$$\lambda \in S^L_{a_0,\theta_0}(M) = \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0, \mu \neq a_0\}$$

inclusion $\lambda^\alpha \in \rho^L(M)$ is valid.

(II) for every $a > a_0$, $\theta \in (\pi/2, \theta_0)$ there exists such constant $K = K(a,\theta) > 0$ that for all $\mu \in S^L_{a,\theta}(M)$ we have

$$\max\left\{\|(\mu^\alpha L - M)^{-1} L\|_{\mathscr{L}(\mathfrak{U})}, \|L(\mu^\alpha L - M)^{-1}\|_{\mathscr{L}(\mathfrak{V})}\right\} \leq \frac{K(a,\theta)}{|\mu^{\alpha-1}(\mu - a)|}.$$

Denote $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$, $\overline{\mathbb{R}}_+ = \{t \in \mathbb{R} : t \geq 0\}$, $g_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ for $\beta > 0$, $t > 0$. A function $u \in C(\mathbb{R}_+; D_L) \cap C(\mathbb{R}_+; D_M)$, such that $Lu \in C^{m-1}(\overline{\mathbb{R}}_+; \mathfrak{U})$, and

$$g_{m-\alpha} * \left(Lu - \sum_{k=0}^{m-1} (Lu)^{(k)}(0) g_{k+1}\right) \in C^m(\mathbb{R}_+; \mathfrak{V}),$$

is called as a solution of equation (1), if for all $t > 0$ equality (1) is valid.

Denote by $\mathfrak{U}^1$ (or $\mathfrak{V}^1$) a closure in $\mathfrak{U}$ (or $\mathfrak{V}$) of the image $\mathrm{im}(\mu L - M)^{-1} L$ (or $\mathrm{im} L(\mu L - M)^{-1}$) for $\mu \in \rho^L(M)$ and by $\mathfrak{U}^0$ (or $\mathfrak{V}^0$) a kernel $\ker L$ (or $\ker L(\mu L - M)^{-1}$). The restriction of the operator $L$ (or $M$) on $D_L \cap \mathfrak{U}^k$ (or $D_M \cap \mathfrak{V}^k$) will be denoted by $L_k$ (or $M_k$), $k = 0, 1$,

**Theorem 1** *Let $\alpha > 0$, (I), (II) be satisfied, $\gamma = \partial S^L_{a_0,\theta_0}(M) + 1$, $\Sigma_{\theta_0} = \{\tau \in \mathbb{C} : |\arg \tau| < \theta_0 - \pi/2, \tau \neq 0\}$, then*

$$\left\{ U_\alpha(\tau) = \frac{1}{2\pi i} \int_\gamma \mu^{\alpha-1} (\mu^\alpha L - M)^{-1} L e^{\mu t} d\mu \in \mathscr{L}(\mathfrak{U}) : \tau \in \Sigma_{\theta_0} \right\}$$

*is an analytic family of operators and for every $a > a_0$, $\theta \in (\pi/2, \theta_0)$ there exists such $C = C(a,\theta)$, that for all $\tau \in \Sigma_\theta$, $n \in \mathbb{N} \cup \{0\}$*

$$\|U^{(n)}_\alpha(\tau)\|_{\mathscr{L}(\mathfrak{U})} \leq \frac{C(a,\theta) e^{a \mathrm{Re} \tau}}{\tau^n},$$

$\ker L \subset \ker U_\alpha(\tau)$, $\mathrm{im} U_\alpha(\tau) \subset \mathfrak{U}^1$ *for every* $\tau \in \Sigma_{\theta_0}$.

*Besides, if Banach spaces $\mathfrak{U}$ and $\mathfrak{V}$ are reflexive, then $\mathfrak{U} = \mathfrak{U}^1 \oplus \mathfrak{U}^0$, $\mathfrak{V} = \mathfrak{V}^1 \oplus \mathfrak{V}^0$.*

*Let operator $L_1^{-1}$ and at least one of operators $L_1$ or $M_1$ be bounded. Then for all $u_k \in D_M \cap \mathfrak{U}^1$ the function $u(t) = \sum_{k=0}^{m-1} J^k_t U_\alpha(t) u_k$ is an unique solution of problem* (1), (2). *If there exists $l \in \{0, 1, \ldots, m-1\}$ such that $u_l \notin \mathfrak{U}^1$ then there is no solution of problem* (1), (2).

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